THE INVERSION INTEGRAL OF THE LAPLACE TRANSFORM

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THE LAPLACE TRANSFORM AND THE OPERATIONAL CALCULUS

The Laplace transform is an integral transformation or operator that transforms a large class of functions, $\{f(t)\}$ $t > 0$, into a class of functions, $\{F(s)\}$, over the complex field. The form of the transform is given by

$$(1.1) \qquad \mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt = F(s), \mathcal{R}[s] > x_0$$

where $\mathcal{R}[s]$ denotes the real part of s and x_0 is a real constant, the value of which depends upon the order properties of f(t).[1] It will be shown that the Laplace integral will converge when the inequality is satisfied. The function

$$(1.2) \qquad f(t) = \mathcal{L}^{-1}\{F(s)\}$$

is said to be the inverse Laplace transform of F(s). It is unique to the extent that two functions having the same transform may differ only at isolated points. The purpose of this paper is to obtain the inversion integral as the general expression for the inverse transform.

The Laplace transform is the basis of the modern operational calculus which is a system for solving certain

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[1]There are other forms of the Laplace transform given in the literature such as: the p-multiplied Laplace transform [10], [11], [13], and the bilateral or two sided Laplace transform [9], [13], [14], [15].

classes of ordinary and partial differential equations, and integrodifferential equations and evaluating improper integrals. These classes include many equations of physical interest. The man credited with the invention of the operational calculus was Oliver Heaviside, an electrical engineer. The operational calculus presented by Heaviside did not involve the use of the Laplace transform, but he did indicate later that his operators could be obtained by its use [13, p. 3]. Heaviside presented his operational calculus without proof or sufficient indication as to how his "ingenious" methods could be put on a sound mathematical basis. This was characteristic of Heaviside whose approach to mathematics was pragmatic or intuitive. He expressed in his writing the feeling that rigorous proofs were not only time consuming, but silly: "The best possible of all proofs is to set out a fact descriptively so that it can be seen to be a fact" [1, p. 207]. He asserted that mathematicians deliberately complicated things by inventing difficulties. Another interesting quotation given by Berg [1, p. 207] is "Physics is above mathematics and the slave must be trained to suit the master's convenience." These out of context comments probably overstate Heaviside's views but they represent an attitude that generated considerable antagonism toward him and his work. Unfortunately, this antagonism and Heaviside's deficiencies in mathematical rigor were probably responsible for the quarter of a century

delay in the wide acceptance of the operational calculus. It was only after others had developed the theory with rigor that the system came into wide usage.

It was about 1916 before T. J. I'A. Bromwich succeeded in breaching the mathematical obscurity of Heaviside's techniques. This prominent mathematican introduced the use of contour integration in the complex plane into the problem. Bromwich not only added rigor, but greatly extended the applicability of the system [2, p. xiv-xv] [13, p. 4]. J. R. Carson introduced the p-multiplied Laplace transform

$$\emptyset\ (p) = p \int_0^{\infty} e^{-pt}\ f(t)\ dt$$

into the theory in 1926. This work contained mathematical deficiencies that were somewhat corrected by van der Pol in 1929, [2, p. xv]. It is interesting to note that van der Pol [13, p. 4] credits P. Levy with the derivation of a form of the inversion integral. This resulted in the synthesis of the Laplace transform approach with the work of Bromwich.

About this same time G. Doetsch was doing much of the same work; according to Carslaw and Jaeger, it was he who first recognized the great significance of the inversion integral and presented the theory of the operational calculus in its present form [2, p. xvi].

To demonstrate briefly the operational methods and relate these to conventional techniques in solving

differential equations, a relatively simple differential equation with associated initial conditions will be considered:

(1.3-a) $$\frac{d^3y}{dt^3} + 3\,\frac{d^2y}{dt^2} + 2\,\frac{dy}{dt} = \sin t,$$

(1.3-b) $$y(0) = y'(0) = y''(0) = 0.$$

The method of undetermined coefficients will be used to solve this system. The complementary solution is

(1.4) $$y_c = C_1 + C_2\,e^{-t} + C_3\,e^{-2t}$$

and a particular solution is of the form

$$y_p = A \cos t + B \sin t.$$

After substituting this into equation 1.3-a and solving for A and B it may be shown that

(1.5) $$y_p = -\frac{1}{10}\cos t - \frac{3}{10}\sin t.$$

Thus the general solution is given by

(1.6) $$y = y_c + y_p = C_1 + C_2\,e^{-t} + C_3\,e^{-2t} - \frac{1}{10}\cos t - \frac{3}{10}\sin t.$$

To obtain the final solution one must apply the initial conditions to equation 1.6 and its first two derivatives, then solve the equations simultaneously to evaluate the

arbitrary constants. Thus, the solution is

$$(1.7) \qquad y = -\frac{1}{10}\cos t - \frac{3}{10}\sin t - \frac{1}{2}e^{-t} + \frac{1}{10}e^{-2t}.$$

This example was chosen for its simplicity but it should suffice to recall to the reader the tedium encountered in the use of the conventional methods that are usually a part of a first course in differential equations. With this thought in mind again consider equation 1.3-a. This time multiply each term by e^{-st} dt and formally integrate over the range zero to infinity without considering what properties must be attributed to y for these integrals to exist. Then,

$$(1.8) \qquad \int_0^\infty e^{-st}y'''(t)\,dt + 3\int_0^\infty e^{-st}y''(t)\,dt$$

$$+ 2\int_0^\infty e^{-st}y'(t)\,dt = \int_0^\infty e^{-st}\sin t\,dt.$$

Let $u = e^{-st}$ and $dv = y'''(t)$ or $y''(t)$ or $y'(t)$ dt, successively, then on integrating by parts, there results

$$(1.9) \qquad (s^3 + 3s^2 + 2s)\int_0^\infty e^{-st}\,y(t)\,dt - \Big[(s^2 + 3s + 2)\,y(0)$$

$$+ (s + 3)\,y'(0) + y''(0)\Big] = \frac{1}{s^2 + 1}.$$

Utilizing equation 1.3-b, equation 1.9 becomes

$$(s^3 + 3s^2 + 2s)\int_0^\infty e^{-st}\,y(t)\,dt = \frac{1}{s^2 + 1}$$

or

$$\int_0^{\infty} e^{-st}\, y(t)\, dt = \frac{1}{(s^2 + 1)(s)(s^2 + 3s + 2)} . \tag{1.10}$$

The differential equation and its associated initial conditions have now been transformed into an integral equation. This would be of little interest unless a means could be found for solving such equations in a direct and simple manner. Note that if the right hand side is decomposed into partial fractions then,

$$\int_0^{\infty} e^{-st}\, y(t)\, dt \tag{1.11}$$

$$= -\frac{1}{10}\left(\frac{s}{s^2 + 1}\right) - \frac{3}{10}\left(\frac{1}{s^2 + 1}\right) + \frac{1}{2}\left(\frac{1}{s}\right) - \frac{1}{2}\left(\frac{1}{s + 1}\right) + \frac{1}{10}\left(\frac{1}{s + 2}\right).$$

If this result is compared with the solution obtained previously, a correspondence between the two sets of coefficients in equations 1.11 and 1.7 is noted. Finally if equation 1.7 is multiplied through by e^{-st} and integrated the result is equation 1.11. The technique just given is essentially the method of the Laplace transform. As presented here it would seem to be of little advantage, but once a relatively small number of transforms are catalogued and theorems of composition are established this method becomes a powerful tool. In practice transform tables would have been used to write

$$s^3Y(s) + 3s^2Y(s) + 2sY(s) = \frac{1}{s^2 + 1}$$

immediately from equations 1.3-a and 1.3-b. This then would be solved for Y(s) and the result decomposed by partial fractions to obtain

$$y(t) = \mathcal{L}^{-1}\, Y(s) = -\frac{1}{10}\mathcal{L}^{-1}\left\{\frac{s}{s^2+1}\right\} - \frac{3}{10}\mathcal{L}^{-1}\left\{\frac{s}{s^2+1}\right\}$$

$$+\frac{1}{2}\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} - \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\} + \frac{1}{10}\mathcal{L}^{-1}\left\{\frac{1}{s+2}\right\}.$$

The solution is then obtained from tables,

$$y(t) = -\frac{1}{10}\cos t - \frac{3}{10}\sin t + \frac{1}{2} - \frac{1}{2}e^{-t} + \frac{1}{10}e^{-2t}.$$

The linearity property of the Laplace transform, used above, follows directly from the linearity of integration. The tedium of the partial fraction decomposition is reduced by formal techniques that are developed in such treatments as Churchill [3, pp. 57-67]. Admittedly, the presentation here was oversimplified; such important questions as the existence of transforms and inverse transforms were completely disregarded. Some of these questions will be answered in terms of theorems in the development of the inversion integral in the next section.

THE THEORY OF THE LAPLACE TRANSFORM AND ITS INVERSION INTEGRAL

The theory of the Laplace transform and its inversion integral will be developed using a form of the Fourier integral theorem as a basis. Thus, the first major concern will be proving that theorem. The proof will require some definitions and theorems which will be stated without proof.

Definitions, Theorems and Notation

The notation $[a,b]$ will denote the closed interval from a to b and (a,b) will denote the open interval. Similarly $(a,b]$ and $[a,b)$ will be used for intervals open on one end and closed on the other. Right hand and left hand limits at any point a will be written as $f(a + 0)$ and $f(a - 0)$ respectively. If $\{A_n\}$ $n = 1,2,\ldots, N$ is a class of numbers, Max $[A_n]$ will signify a number of the set which is greater than or equal to all the other members of the set; similarly Min $[A_n]$ will indicate a number of the set that is less than or equal to all other members of the set. Unless otherwise indicated the functions and variables in the following definitions and theorems will be real.

Definition 2.1.1. The function $f(x)$ is piecewise continuous on a finite closed interval $[a,b]$ if it is

bounded and is discontinuous at only a finite number of isolated points. The notation f(x) is PC will be used to indicate that the function f(x) is piecewise continuous in every finite interval.

It follows from the definition that PC functions will have left and right hand limits at all points and that each of the discontinuities may be enclosed in a sufficiently small interval such that the sum of their lengths may be made arbitrarily small. With the aid of the following definition a less restricted class of functions may be defined.

Definition 2.1.2. The function f(x) is of order g(x) as x tends to a if there exists a finite number M such that

$$\left|f(x)\right| \leqslant M\left|g(x)\right|$$

throughout some neighborhood of a. This will be expressed by the following notation: $f(x) = 0\left[g(x)\right]$.

Definition 2.1.3. The function f(x) is almost piecewise continuous on the closed interval $\left[a,b\right]$ if it is piecewise continuous except for singularities at a finite number of points $\left\{x_k\right\}$, $k = 1,2,\ldots,N$, where $f(x) = 0\left[|x - x_k|^{-n}\right]$, $n<1$ as x tends to x_k. The notation, f(x) is APC will indicate that the function f(x) is almost piecewise continuous on every finite interval.

This class will admit functions with certain infinite discontinuities that are integrable in any finite interval

since the order property guarantees the convergence of such improper integrals.

The following theorem is referred to as Cauchy's principle for the convergence of improper integrals.

Theorem 2.1.1. a) The integral $\int_a^\infty f(x)\,dx$ converges if and only if for every $\epsilon > 0$ there exists an X such that

$$\left|\int_A^{A'} f(x)\,dx\right| < \epsilon \text{ when } A,\ A' > X.$$

b) If f(x) has a singular point at x = c then the integral $\int_a^c f(x)\,dx$, $a < c$ converges if and only if for every $\epsilon > 0$ there exists a $\delta > 0$ such that

$$\left|\int_{c'}^{c} f(x)\,dx\right| < \epsilon \text{ when } 0 < c - c' < \delta .$$

Definition 2.1.4. The principal value of the integral $\int_{-\infty}^{\infty} f(x)\,dx$, denoted PV$\int_{-\infty}^{\infty} f(x)\,dx$, is defined as

$$\lim_{A\to\infty}\int_{-A}^{A} f(x)\,dx$$

if that limit exists.

In the proof of the Fourier integral theorem it will be necessary to reverse the order of integration of iterated integrals with infinite limits and certain allowable discontinuities. The necessary justification will be found in the next two theorems. Since these theorems involve the concept of uniform convergence, it will be defined before stating the theorems.

Definition 2.1.5. The integral $\int_a^\infty f(x,y)\,dx$ converges

uniformly with respect to y at the upper limit, for y in the interval $c \leqslant y \leqslant d$, if for every $\epsilon > 0$ there exists an $X(\epsilon)$ independent of y such that

$$\left| \int_A^{A'} f(x,y)\, dx \right| < \epsilon \text{ for } A,\ A' > X$$

and for all y in $[c,d]$.

<u>Theorem 2.1.2</u>. If $f(x,y)$ is APC for x and y in the respective intervals $a \leqslant x \leqslant b$ and $c \leqslant y \leqslant d$ except possibly at a finite number of isolated points, then

$$\int_c^d dy \int_a^b f(x,y)\, dx = \int_a^b dx \int_c^d f(x,y)\, dy.$$

For proof see LePage [9, pp. 242-46].

<u>Theorem 2.1.3</u>. If $f(x,y)$ is APC in x and y except for at most a finite number of points in every finite rectangle in the xy-plane and if the integral $\int_a^\infty f(x,y)\, dx$ converges uniformly with respect to y at the upper limit, for y in $[c,d]$, then

$$\int_c^d dy \int_a^\infty f(x,y)\, dx = \int_a^\infty dx \int_c^d f(x,y)\, dy.$$

For proof see LePage [9, pp. 246-49].

Although the following theorem is not used in the proof of the Fourier integral theorem it will be needed later. It is given here because of its similarity to the two theorems just given.

<u>Theorem 2.1.4</u>. If $f(x,z)$, z complex, is APC in x for a fixed z and continuous in z for each x where the function

is defined and if $\int_a^\infty f(x,z)\,dx$ is uniformly convergent with respect to z at the upper limit for z on the curve C, then

$$\int_C dz \int_a^\infty f(x,z)\,dx = \int_a^\infty dx \int_C f(x,z)\,dz.$$

For proof see LePage [9, pp. 249-50].

The Weierstrass test for uniform convergence of improper integrals is given by the following theorem.

Theorem 2.1.5. If M(x) is a positive function such that $|f(x,z)| < M(x)$ for all z in a region R, z complex, and if $\int_a^\infty M(x)\,dx$ exists then the integral $\int_a^\infty f(x,z)\,dx$ converges uniformly with respect to z at the upper limit for all z in R.

The Lipschitz conditions will be needed in the statement of the Fourier integral theorem [9, p. 271].

Definition 2.1.6. The function f(x) satisfies the Lipschitz conditions of order α for u in the interval $[0, \delta]$ if there exist numbers K_1, K_2, α_1, and α_2 such that

$$|f(t - u) - f(t - 0)| \leqslant K_1 u^{\alpha_1}$$

and

$$|f(t + u) + f(t + 0)| \leqslant K_2 u^{\alpha_2}$$

when $0 \leqslant u \leqslant \delta$ and $\alpha = \text{Min}\,[\alpha_1, \alpha_2]$.

Proof of the Fourier Integral Theorem

The first steps in the proof of the Fourier integral theorem will be the proofs of two lemmas which together are

a weakened form of the Riemann-Lebesgue theorem for trigonometric integrals.

<u>Lemma</u> <u>2.2.1</u>. If f(x) is PC on $[a,b]$ then,

$$\lim_{|y|\to\infty}\int_a^b f(x)\ \sin yx\ dx = 0$$

and

$$\lim_{|y|\to\infty}\int_a^b f(x)\ \cos yx\ dx = 0.$$

<u>Proof</u>. Since f(x) is PC, for any $\epsilon > 0$ one can define

$$g(x) = \begin{cases} A_0 & a < x < x_1 \\ A_1 & x_1 < x < x_2 \\ \cdots\cdots\cdots\cdots & \\ A_{N-1} & x_{N-1} < x < b \end{cases}$$

where the $\{A_k\}$ k = 0,1,2,..., N - 1 are constants such that

$$\int_a^b |f(x) - g(x)|\ dx < \frac{\epsilon}{2}.$$

Now consider the difference,

$$\left|\int_a^b f(x)\ \sin yx\ dx\right| - \left|\int_a^b g(x)\ \sin yx\ dx\right|$$

$$\leqslant \left|\int_a^b [f(x) - g(x)]\ \sin yx\ dx\right| \leqslant \int_a^b |f(x) - g(x)|\ dx < \frac{\epsilon}{2}.$$

Hence

$$\left|\int_a^b f(x)\ \sin yx\ dx\right| < \frac{\epsilon}{2} + \left|\int_a^b g(x)\ \sin yx\ dx\right|$$

where

$$\left|\int_a^b g(x)\ \sin yx\ dx\right| = \left|\sum_{k=1}^{N} A_k \int_{x_k}^{x_{k+1}} \sin yx\ dx\right|$$

$$= \left| \sum_{K=1}^{N} A_k \frac{\cos yx_k - \cos yx_{k+1}}{y} \right| \leq \frac{M}{|y|} \sum_{K=1}^{N} \left[|\cos yx_{k+1}| + |\cos yx_k| \right]$$

$$\leq \frac{2NM}{|y|}, \; M = \text{Max} \left[|A_k| \right].$$

Thus, if $|y| > \frac{4NM}{\epsilon}$, then $\frac{2NM}{|y|} < \frac{\epsilon}{2}$ and $\left| \int_a^b f(x) \sin yx \, dx \right| < \epsilon$.

Since one can replace sin yx by cos yx without affecting the validity of the proof and since ϵ can be made arbitrarily small the lemma has been proven.

The next lemma will generalize the results just obtained.

Lemma 2.2.2. If $f(x)$ is APC for all x and if $\int_0^\infty |f(x)| dx$ exists, then

$$\lim_{|y| \to \infty} \int_a^\infty f(x) \sin yx \, dx = 0 \qquad a > 0$$

and

$$\lim_{|y| \to \infty} \int_a^\infty f(x) \cos yx \, dx = 0 \qquad a > 0.$$

Proof. Since $\int_a^\infty |f(x)| dx$ exists, if x_o is a singular point of $f(x)$ then there exist three numbers X_1, X_2 and X_3 such that $X_1 < x_o < X_2 < X_3$ and

$$\left| \int_{X_1}^{X_2} f(x) \sin yx \, dx \right| < \int_{X_1}^{X_2} |f(x)| dx < \frac{\epsilon}{4}$$

and

$$\left| \int_{X_3}^{\infty} f(x) \sin yx \, dx \right| < \int_{X_3}^{\infty} |f(x)| dx < \frac{\epsilon}{4}.$$

By lemma 2.2.1 there exist y_1 and y_2 such that

$$\left| \int_0^{X_1} f(x) \sin yx \, dx \right| < \frac{\epsilon}{4} \text{ for } |y| > |y_1|$$

and

$$\left| \int_{X_2}^{X_3} f(x) \sin yx \, dx \right| < \frac{\epsilon}{4} \quad \text{for } |y| > |y_2| .$$

Hence

$$\left| \int_{o}^{\infty} f(x) \sin yx \, dx \right| < \epsilon \quad \text{for } |y| > |y_o|$$

where

$$y_o = \text{Max} \left[y_1, y_2 \right] .$$

The second statement of the lemma may be proven in a similar manner by replacing sin yx by cos yx; thus the proof is complete.

The following form of the Fourier integral theorem will be used.

Theorem 2.2.1. If f(t) is APC and satisfies the Lipschitz conditions of order $\alpha > 0$ at points where f(t) is finite and if $\int_{-\infty}^{\infty} |f(t)| \, dt$ converges, then

$$\mathfrak{F}(iy) = \int_{-\infty}^{\infty} f(t) \, e^{-iyt} \, dt \tag{2.2.1}$$

converges uniformly with respect to y at the infinite limits, thus defining the function $\mathfrak{F}(iy)$ for all real y. Also,

$$\frac{f(t+0) + f(t-0)}{2} = \frac{1}{2\pi} \, \text{PV} \int_{-\infty}^{\infty} \mathfrak{F}(iy) \, e^{iyt} \, dy \tag{2.2.2}$$

at all points where f(t) is finite.

Proof. Consider

$$\left| \int_{-\infty}^{\infty} f(t) \, e^{-iyt} \, dt \right| \leq \int_{-\infty}^{\infty} \left| f(t) \, e^{-iyt} \right| dt \leq \int_{-\infty}^{\infty} |f(t)| \, dt.$$

The uniform convergence follows from Theorem 2.1.5. Substitute the right side of equation 2.2.1 into the integral in equation 2.2.2, then

$$\mathrm{PV}\int_{-\infty}^{\infty} \mathcal{F}(iy)\, e^{iyt}\, dy = \lim_{A\to\infty}\int_{-A}^{A} e^{iyt}\, dy \int_{-\infty}^{\infty} f(\tau)\, e^{-iy\tau}\, d\tau$$

$$= \lim_{A\to\infty}\int_{-A}^{A} dy \int_{-\infty}^{\infty} f(\tau)\, e^{iy(t-\tau)}\, d\tau.$$

Theorem 2.1.3 applies since the integral in equation 2.2.1 is uniformly convergent, hence

$$\mathrm{PV}\int_{-\infty}^{\infty} \mathcal{F}(iy)\, e^{iyt}\, dy = \lim_{A\to\infty}\int_{-\infty}^{\infty} f(\tau)\, d\tau \int_{-A}^{A} e^{i(t-\tau)y}\, dy$$

$$= \lim_{A\to\infty}\int_{-\infty}^{\infty} f(\tau)\, d\tau \left[\int_{-A}^{A} \cos\,(t-\tau)\, y\, dy + i\int_{-A}^{A} \sin\,(t-\tau)\, y\, dy\right]$$

$$= 2\lim_{A\to\infty}\int_{-\infty}^{\infty} f(\tau)\, \frac{\sin\,(t-\tau)\,A}{t-\tau}\, d\tau.$$

Make the change of variable $u = \tau - t$, considering t constant, then

$$\mathrm{PV}\int_{-\infty}^{\infty} \mathcal{F}(iy)\, e^{iyt}\, dy = 2\lim_{A\to\infty}\int_{-\infty}^{\infty} f(u+t)\, \frac{\sin uA}{u}\, du.$$

Now for any t for which $f(t-0)$ and $f(t+0)$ exist choose a $\delta > t$, then

$$2\int_{-\infty}^{\infty} f(u+t)\, \frac{\sin Au}{u}\, du = I_1(A,t) + I_2(A,t) + I_3(A,t)$$

where

$$I_1(A,t) = 2\int_{-\infty}^{-\delta} f(u+t)\, \frac{\sin Au}{u}\, du$$

$$I_2(A,t) = 2\int_{-\delta}^{\delta} f(u + t) \frac{\sin Au}{u} du$$

$$I_3(A,t) = 2\int_{\delta}^{\infty} f(u + t) \frac{\sin Au}{u} du.$$

By lemma 2.2.2 $I_1(A,t)$ and $I_3(A,t)$ will vanish as A increases without bound.

Therefore,

$$PV\int_{-\infty}^{\infty} \mathcal{F}(iy)\, e^{iyt}\, dy$$

$$= 2 \lim_{A\to\infty}\left[\int_{-\delta}^{0} f(u + t) \frac{\sin Au}{u} du + \int_{0}^{\delta} f(u + t) \frac{\sin Au}{u} du\right]$$

$$= 2 \lim_{A\to\infty}\left[\int_{0}^{\delta} f(t - u) \frac{\sin Au}{u} du + \int_{0}^{\delta} f(t + u) \frac{\sin Au}{u} du\right].$$

Thus

(2.2.3) $$PV\int_{-\infty}^{\infty} \mathcal{F}(iy)\, e^{iyt}\, dy$$

$$= 2 \lim_{A\to\infty} \int_{0}^{\delta} \left[f(t + u) + f(t - u)\right] \frac{\sin Au}{u} du.$$

Now consider

$$2\left[f(t + 0) + f(t - 0)\right]\int_{0}^{\infty} \frac{\sin w}{w} dw = \pi\left[f(t + 0) + f(t - 0)\right].$$

Also

$$\int_{0}^{\infty} \frac{\sin w}{w} dw = \lim_{A\to\infty}\int_{0}^{A\delta} \frac{\sin w}{w} dw = \lim_{A\to\infty}\int_{0}^{\delta} \frac{\sin Au}{u} du.$$

Hence

(2.2.4) $$2\left[f(t + 0) + f(t - 0)\right]\int_{0}^{\infty} \frac{\sin w}{w} dw$$

$$= \lim_{A\to\infty} 2\int_{0}^{\delta} \left[f(t + 0) + f(t - 0)\right] \frac{\sin Au}{u} du.$$

Now consider,

$$\lim_{A\to\infty}\left\{\int_0^{\delta}\frac{f(t+u)+f(t-u)}{u}\sin Au\,du - \int_0^{\delta}\frac{f(t+0)+f(t-0)}{u}\sin Au\,du\right\}$$

or

$$(2.2.5)\quad \lim_{A\to\infty}\left\{\int_0^{\delta}\frac{f(t+u)-f(t+0)}{u}\sin Au\,du + \int_0^{\delta}\frac{f(t-u)-f(t-0)}{u}\sin Au\,du\right\}$$

which is the difference of the right members of equations 2.2.4 and 2.2.3. Since f(t) satisfies the Lipschitz conditions $\alpha > 0$, then

$$\left|\int_0^{\delta}\frac{f(t+u)-f(t+0)}{u}\,du\right| \leqslant \int_0^{\delta} K_1 u^{\alpha_1-1}du = \frac{K_1\delta^{\alpha_1}}{\alpha_1}$$

and

$$\left|\int_0^{\delta}\frac{f(t-u)-f(t-0)}{u}\,du\right| \leqslant \int_0^{\delta} K_2 u^{\alpha_2-1}du = \frac{K_2\delta^{\alpha_2}}{\alpha_2}.$$

Since these integrals exist lemma 2.2.2 applies to the integrals in 2.2.5. Thus the expression 2.2.5 vanishes in the limit as A increases without bound.

Hence,

$$PV\int_{-\infty}^{\infty}\mathcal{F}(iy)\,e^{iyt}\,dy = \pi\left[f(t+0)+f(t-0)\right]$$

or

$$\frac{f(t+0)+f(t-0)}{2} = \frac{1}{2\pi}\,PV\int_{-\infty}^{\infty}\mathcal{F}(iy)\,e^{iyt}\,dy$$

at values of t for which f(t) is finite, and

(2.2.5)
$$f(t) = \frac{1}{2\pi} \text{PV} \int_{-\infty}^{\infty} \mathcal{F}(iy)\, e^{iyt}\, dy,$$

where f(t) is continuous. This completes the proof of the Fourier integral theorem.

The function $\mathcal{F}(iy)$ will be referred to as the Fourier transform of f(t) and equation 2.2.5 as its inversion formula. The next question to be considered is that of uniqueness. It follows from the uniqueness of integration that each function f(t) satisfying the conditions of theorem 2.2.1 will have a unique transform. Conversely, the inversion is unique as qualified by the following theorem.

Theorem 2.2.2. If $\mathcal{F}(iy)$ is any function for which an inversion exists, then any two functions f_1 and f_2 for which $\mathcal{F}(iy)$ is the Fourier transform are related by the equation

$$f_1(t) = f_2(t) + N(t) \text{ where } \int_{-\infty}^{\infty} N(t)\, dt = 0.$$

Proof. The functions $f_1(t)$ and $f_2(t)$ have the same transform, hence by theorem 2.2.1

$$\frac{f_1(t + 0) + f_1(t - 0)}{2} = \frac{f_2(t + 0) + f_2(t - 0)}{2}$$

at all finite points of f(t). Hence at all continuous points

$$f_1(t) \equiv f_2(t).$$

Since $f_1(t)$ and $f_2(t)$ are APC they may differ only at a countable number of isolated points; denote these as $\{t_n\}$ $n = 1,2,3,\ldots$. Hence, $f_1(t) = f_2(t) + N(t)$ where

$$N(t) = \left\{ \begin{array}{ll} 0, & t \neq t_n \\ & \\ f_1(t_n) - f_2(t_n), & t = t_n \end{array} \right\} ;$$

$N(t)$ is zero except at a countable number of points, thus

$$\int_{-\infty}^{\infty} N(t)\, dt = 0.$$

The proof is complete.

Now consider the function $f(t)$ such that $f(t) = 0$ for $t < 0$. Recall that in the introduction the Laplace transform was defined as having the form

$$F(s) = \int_{0}^{\infty} f(t)\, e^{-st}\, dt = \int_{-\infty}^{\infty} \left[f(t)\, e^{-xt}\right] e^{-iyt}\, dt$$

$$\text{if } s = x + iy.$$

But this last expression is the Fourier transform of $f(t)\, e^{-xt}$. If the Fourier inversion formula is formally applied on the assumption that the $f(t)\, e^{-xt}$ satisfies the conditions of theorem 2.2.1, then

$$f(t)\, e^{-xt} = \frac{1}{2\pi} \lim_{A \to \infty} \int_{-A}^{A} F(s)\, e^{iyt}\, dy$$

or

$$f(t) = \frac{1}{2\pi} \lim_{A \to \infty} \int_{-A}^{A} F(s)\, e^{st}\, dy$$

where $s = x + iy$. Now consider the contour integral

$$\int_C F(s)\, e^{st}\, ds$$

where C is the vertical line at some fixed abscissa x which is traversed from $y = -A$ to $y = A$. If A is increased without bound, this path sometimes is referred to as the Bromwich contour. This integral may be written

$$\int_{x-iA}^{x+iA} F(s)\, e^{st}\, ds = \int_{-A}^{A} F(s)\, e^{st}\, (idy).$$

Hence

$$f(t) = \lim_{A\to\infty} \frac{1}{2\pi} \int_{-A}^{A} F(s)\, e^{st}\, dy \tag{2.3.1}$$

$$= \lim_{A\to\infty} \frac{1}{2\pi i} \int_{x-iA}^{x+iA} F(s)\, e^{st}\, ds.$$

This last expression is the inversion integral of the Laplace transform. Although the form of the sought after inversion has been obtained it would be of little value with only the limited material thus far presented. It is, in fact, ambiguous unless a means is provided for specifing the value to be assigned to x. One would have little hope of finding a unique f(t) from a given F(s), even if he were assured of its existence. The additional information necessary to implement the use of the inversion formula will be obtained by investigating the properties of the Laplace transform in terms of the theory of functions of a complex variable.

Properties of the Laplace Transform

The first property of the Laplace transform to be investigated will be its convergence. If the conditions of theorem 2.2.1 are to be satisfied then the integral

$$(2.3.1) \qquad \int_0^\infty f(t)\, e^{-st}\, dt = \int_0^\infty \left[f(t)\, e^{-xt}\right] e^{-iyt}\, dt$$

must converge absolutely. For a given $f(t)$ this integral may converge for some values of s and not for others. The form of the expression on the right side of equation 2.3.1 suggests an examination of the class of functions of exponential order. This leads to the following theorem.

Theorem 2.3.1. If $f(t)$ is APC and $O\left[e^{x_0 t}\right]$ then the integral $\int_0^\infty f(t)\, e^{-st}\, dt$, $s = x + iy$, converges absolutely for $x > x_0$ and uniformly with respect to x and y at the upper limit for $x \geqslant x_1 > x_0$.

Proof. Note that

$$\int_0^\infty \left| f(t)\, e^{-st}\right| dt = \int_0^\infty |f(t)|\, e^{-xt}\, dt.$$

Since $f(t)$ is APC, the integral $\int_0^{T_0} |f(t)| e^{-xt}\, dt$ exists for any finite T_0. Also, $f(t)$ is $O\left[e^{x_0 t}\right]$; hence there is a number M such that $|f(t)|\, e^{-xt} \leqslant M\, e^{-(x-x_0)t}$. From this it follows that

$$\lim_{t\to\infty} |f(t)|\, e^{-xt} = 0 \text{ for } x > x_0.$$

This implies that for every $\epsilon_1 > 0$ there is a T_0 such that

$$|f(t)|\, e^{-x_2 t} < \epsilon_1 \quad \text{for } x_0 < x_2 < x \text{ and } t > T_0.$$

Hence

$$(2.3.2) \qquad \int_{T_0}^{\infty} |f(t)| e^{-xt}\, dt = \int_{T_0}^{\infty} |f(t)| e^{-x_2 t} e^{-(x-x_2)t}\, dt$$

$$\leqslant \epsilon_1 \int_{T_0}^{\infty} e^{-(x-x_2)t}\, dt$$

where $\int_{T_0}^{\infty} e^{-(x-x_2)t}\, dt$ exists for $x > x_2$. Thus

$$\int_0^{\infty} |f(t)\, e^{-st}|\, dt \leqslant \int_0^{T_0} |f(t)\, e^{-st}|\, dt$$

$$+ \epsilon_1 \int_{T_0}^{\infty} e^{-(x-x_2)t}\, dt.$$

Since both integrals in the right hand member have been shown to exist the absolute convergence has been proven. Now choose x_1, such that $x \geqslant x_1 > x_2 > x_0$. Then from equation 2.3.2

$$\int_{T_0}^{\infty} |f(t)| e^{-xt}\, dt < \epsilon_1 \int_{T_0}^{\infty} e^{-(x-x_2)t}\, dt < \epsilon_1 \int_{T_0}^{\infty} e^{-(x_1 - x_2)t}\, dt.$$

This last expression on the right is independent of x or y and the uniform convergence follows from theorem 2.1.5. The proof is complete.

Corollary 2.3.1. If $f(t)$ is APC and is identically zero for $t > T_0$, where T_0 is any positive number, then $F(s)$ converges for all s.

Unless otherwise indicated the remainder of this paper will be restricted to APC functions of exponential order where the function is identically zero for $t < 0$.

Theorem 2.3.1 has established a half-plane of uniform convergence for functions of exponential order. The value x_o given in theorem 2.3.1 which fixes the boundary of the region of convergence of the Laplace integral is called the abscissa of convergence and the vertical line through x_o that forms the boundary is the axis of convergence.

The next question to be considered is the uniform convergence at allowable singular points of f(t).

Theorem 2.3.2. If f(t) is APC and if t_k is a singular point, then $\int_o^\infty f(t)\, e^{-st}\, dt$ converges uniformly with respect to s at t_k for s in any right half-plane.

Proof. The function f(t) is APC thus the integral

$$\int_{t_k-\delta_1}^{t_k+\delta_2} |f(t)|\,dt$$

exists where δ_1 and δ_2 are arbitrarily small numbers. If x_1 is any real number, then

$$\int_{t_k-\delta_1}^{t_k+\delta_2} |f(t)|\, e^{|x_1|t}\, dt \leqslant e^{|x_1|(t_k+\delta_2)} \int_{t_k-\delta_1}^{t_k+\delta_2} |f(t)|\,dt$$

also converges. If $x \geqslant x_1$ then,

$$|f(t)\, e^{-st}| = |f(t)|\, e^{-xt} \leqslant |f(t)|\, e^{|x_1|t}.$$

Hence

$$\int_{t_k-\delta_1}^{t_k+\delta_2} |f(t)|\, e^{-st}\, dt \leqslant e^{|x_1|(t_k+\delta_2)} \int_{t_k-\delta_1}^{t_k+\delta_2} |f(t)|\,dt.$$

Therefore, the uniform convergence for $x \geqslant x_1$ follows from theorem 2.1.5. But x_1 was any real number, thus the theorem is proven.

The theorems just given have established a region of convergence for the Laplace integral in the complex plane for a large class of functions. The function

$$F(s) = \int_0^\infty f(t)\, e^{-st}\, dt \text{ for } x > x_0 \text{ if } s = x + iy$$

will now be investigated with respect to the theory of functions of a complex variable for s in the region of convergence of the Laplace integral.

Definition 2.3.1. A function f(s), s complex, is holomorphic at a point if it is differentiable throughout some neighborhood of that point. The function is holomorphic in a domain if it is holomorphic at each point of the domain. If a function is holomorphic in any region of the complex plane, then that function and its analytic continuation will be referred to as an analytic function.

Definition 2.3.2. A function f(s) is a real function of a complex variable if it is real when s is real. The notations $\mathcal{R}[s]$ and $\mathcal{J}[s]$ will denote the real and imaginary parts of the variable s, respectively. The following two theorems will be given without proof.

Theorem 2.3.3. (Cauchy's Theorem) If f(s), s = x + iy, is holomorphic throughout a simply connected domain D, then for every simple closed rectifiable oriented contour C lying on the interior of D

$$\oint_C f(s)\, ds = 0.$$

For proof see [7, pp. 163-68].

Theorem 2.3.4. (Morera's Theorem) If a function f(s) is continuous throughout a simply connected domain D and if for every closed rectifiable contour C interior to D

$$\oint_C f(s)\, ds = 0,$$

then f(s) is holomorphic throughout the interior of D. For proof see [7, pp. 188-89].

These two theorems will now be used to prove the following theorem.

Theorem 2.3.5. The Laplace transform of f(t),

$$F(s) = \int_0^\infty f(t)\, e^{-st}\, dt \text{ for } x > x_0 \text{ where } s = x + iy,$$

is holomorphic interior to the region of convergence of the integral.

Proof. Consider any simple closed rectifiable positively oriented curve C in the region of convergence of the integral. Then

$$\oint_C F(s)\, ds = \oint_C ds \int_0^\infty f(t)\, e^{-st}\, dt.$$

The conditions of theorem 2.1.4 are satisfied, hence the order of integration may be reversed. Then, by theorem 2.3.3

$$\oint_C F(s)\, ds = \int_0^\infty f(t)\, dt \oint_C e^{-st}\, ds = 0$$

since e^{-st} is an entire function. Since $\oint_C F(s)\, ds = 0$, it follows from theorem 2.3.4 that F(s) is holomorphic throughout the interior of the region of convergence of the Laplace integral. The proof is complete.

The first consequence of theorem 2.3.5 is that the function obtained from integrating the Laplace integral may exist outside the region of convergence of the integral. Since it has been shown that the integral defines a holomorphic function within its region of convergence it follows that it may be possible to continue analytically the function to the entire plane or to a natural boundary. The function F(s) will henceforth denote the analytic function defined by the Laplace integral and its analytic continuation. The function obtained by integrating the Laplace integral will be a global definition of F(s). It follows from the proof of theorem 2.3.1 that the Laplace integral must fail to exist at $s = x_0$ on the axis of convergence. Hence, there will be a singular point of F(s) on this axis. Also any vertical line with $x = x_1$ and $x_1 > x_0$ will lie to the right of all singular points of F(s).

The next consequence of theorem 2.3.5 is given in the form of a theorem.

<u>Theorem</u> <u>2.3.6</u>. If f(t) is a real function of t and if F(s) is a single valued function, then F(s) is a real valued function of the complex variable s.

<u>Proof</u>. Let $s = x + iy$ and consider $\int_0^\infty f(t)\, e^{-st}\, dt$ which converges and is holomorphic in the half-plane where $x > x_0$. The functions f(t) and e^{-xt} are real, hence the integrand is real for $s = x$, $x > x_0$. Therefore, the integral is real for s on the x axis beyond the abscissa of convergence. The proof of the theorem follows from the reflection principle:

The reflection principle. If a function f(s) is holomorphic in some domain that is symmetric to the real axis and if f(x) is real whenever x is a point on that segment then $f(\bar{s}) = \overline{f(s)}$ which implies that f(s) is a real function of the complex variable s. For referrence see [3, p. 265].

The utility of this theorem is derived from the permanence of forms. The Laplace integral may be evaluated by treating s as a real variable to obtain results that are valid when the real variable is replaced by a complex variable. This result is due to a theorem from the theory of functions of a complex variable which states that the form of any function in its domain of holomorphy is uniquely determined by its values along any curve or in any sub-domain.

The questions concerning the inversion integral that were previously noted on page 21 will be answered for APC functions of exponential order by the next theorem and its corollary.

Theorem 2.3.7. If F(s) is any real function of the complex variable $s = x + iy$ that is holomorphic in a half-plane $x > x_0$ and if F(s) is $O[s^{-k}]$, $k > 0$, then

$$f(t) = \frac{1}{2\pi i} \lim_{A\to\infty} \int_{x-iA}^{x+iA} F(s)\, e^{st}\, ds$$

is independent of x for $x > x_0$.

Proof. Let γ and γ_1 be two real numbers such that $x_0 < \gamma < \gamma_1$. Let β be a real number, then consider the rectangle in the complex plane with vertices at $\gamma + i\beta$, $\gamma - i\beta$, $\gamma_1 - i\beta$ and $\gamma_1 + i\beta$. Let the contour C be the perimeter of

this rectangle traversed in a counter-clockwise direction. According to theorem 2.3.3

$$\oint_C F(s)\, e^{st}\, ds = 0.$$

Thus

$$\int_{\beta}^{-\beta} F(\gamma + iy)\, e^{(\gamma+iy)t}\, dy + \int_{\gamma}^{\gamma_1} F(x - i\beta)\, e^{(x-i\beta)t}\, dx$$

$$+ \int_{-\beta}^{\beta} F(\gamma_1 + iy)\, e^{(\gamma_1+iy)t}\, dy + \int_{\gamma_1}^{\gamma} F(x + i\beta)\, e^{(x+i\beta)t}\, dx = 0.$$

Now consider

$$\int_{\gamma_1}^{\gamma} F(x + i\beta)\, e^{(x+i\beta)t}\, dx. \tag{2.3.3}$$

Since $F(s)$ is $0\left[s^{-k}\right]$, $k > 0$, then $\left|F(s)\right| \leqslant \frac{M}{|s|^k}$. Hence,

$$\left|F(s)\, e^{st}\right| < \frac{e^{xt}\, M}{\beta^k} \text{ where M is a constant.}$$

Thus,

$$\left|\int_{\gamma_1}^{\gamma} F(x + i\beta)\, e^{(x+i\beta)t}\, dx\right| \leqslant \frac{M}{\beta^k}\int_{\gamma_1}^{\gamma} e^{xt}\, dx$$

$$= \frac{-M}{t\beta^k}\left(e^{\gamma_1 t} - e^{\gamma t}\right) \text{ for } k > 0.$$

Hence,

$$\lim_{\beta\to\infty}\int_{\gamma_1}^{\gamma} F(x + i\beta)\, e^{(x+i\beta)t}\, dx = 0.$$

Similarly,

$$\lim_{\beta\to\infty}\int_{\gamma}^{\gamma_1} F(x - i\beta)\, e^{(x-i\beta)t}\, dx = 0.$$

Hence,

$$\lim_{\beta\to\infty}\int_{-\beta}^{\beta} F(\gamma + iy)\, e^{(\gamma+iy)t}\, dy = \lim_{\beta\to\infty}\int_{-\beta}^{\beta} F(\gamma_1 + iy)\, e^{(\gamma_1+iy)t}\, dy.$$

Or equivalently

$$\lim_{\beta\to\infty}\int_{\gamma-i\beta}^{\gamma+i\beta} F(s)\, e^{st}\, ds = \lim_{\beta\to\infty}\int_{\gamma_1-i\beta}^{\gamma_1+i\beta} F(s)\, e^{st}\, ds$$

where these are the line integrals over the vertical lines through γ and γ_1. Since γ and γ_1 are any two values of x satisfying the condition $x_0 < \gamma < \gamma_1$ the theorem is proved.

Corollary 2.3.7. If F(s) is the Laplace transform of the APC function f(t) which is $O\left[e^{x_0 t}\right]$, then

$$\lim_{A\to\infty}\int_{x-iA}^{x+iA} F(s)\, e^{st}\, ds$$

is independent of x for $x > x_0$.

Proof. The function f(t) is APC and $O\left[e^{x_0 t}\right]$, hence by theorem 2.3.1 the integral

$$\int_0^\infty \left|f(t)\, e^{-st}\right| dt = \int_0^\infty \left|f(t)\right| e^{-xt}\, dt$$

exists for $x > x_0$. Now write,

$$F(x+iy) = \int_0^\infty f(t)\, e^{-xt} \cos yt\, dt + i\int_0^\infty f(t)\, e^{-xt} \sin yt\, dt.$$

By lemma 2.2.2,

$$\lim_{|y|\to\infty} F(x+iy) = 0 \text{ for } x > x_0.$$

Now consider the same integral that was considered in theorem 2.3.7. Thus, for any $\epsilon_1 > 0$, $\operatorname{Max}\left[F(x+iy)\right] < \epsilon_1$ for $x_0 < \gamma \leqslant x \leqslant \gamma_1$ if $|y|$ is sufficiently large. Considering the expression 2.3.3,

$$\left|\int_{\gamma_1}^{\gamma} F(x+i\beta)\, e^{(x+i\beta)t}\, dx\right| < \epsilon_1 \int_{\gamma_1}^{\gamma} e^{xt}\, dx < \epsilon$$

for sufficiently large $|y|$. The remainder of the proof of the corollary is identical with that of the theorem.

Thus the preceding theorems and the above corollary have removed the ambiguity in the formal development given for the inversion integral

$$\frac{1}{2\pi i}\lim_{A\to\infty}\int_{x-iA}^{x+A} F(s)\, e^{st}\, ds = f(t), \qquad x > x_0$$

for functions whose transforms are of order s^{-k}, $k > 0$ and for transforms of APC functions of exponential order. The uniqueness of the inversion as specified will follow directly from theorem 2.2.2. That is, two functions having the same Laplace transform may differ at most by a null function.

The Evaluation of the Inversion Integral

It will now be shown that certain functions F(s) may be inverted by using the residue theorem to evaluate the inversion integral. First, some notation will be given. A curve in the complex plane will be designated by

$$C: \left\{ s \,\middle|\, s = \psi(t),\ a \leqslant t \leqslant b \right\}.$$

This is read: C is the set of points determined by the continuous function $\psi(t)$ as t varies from a to b. The variable t is real and the order of the end points will denote the orientation, that is, the initial and terminal points.

As an example

$$C: \left\{ s \middle| s = Re^{i\theta},\ \frac{\pi}{2} \geqslant \theta \geqslant -\frac{\pi}{2} \right\}$$

is the semicircle in the positive half-plane of radius R that is centered at the origin; it is traversed in the clockwise direction. Let s_k be a point and C be a curve in the complex plane, then define

$$d\left[s_k,\ C\right] \equiv \text{Min}\left[\left|s_k - s\right|\right],\ s \text{ on } C.$$

Thus it is the absolute "distance" from the point s_k to the nearest point on the curve.

Now consider,

$$f(t) = \frac{1}{2\pi i} \lim_{A_n \to \infty} \int_{x_1 - iA_n}^{x_1 + iA_n} F(s)\, e^{st}\, ds,\ s = x + iy$$

where F(s) is holomorphic in the half plane $x > x_o$. Assume that all singular points $\left\{s_k\right\}$ k = 1,2,3,... of F(s) and hence of F(s) e^{st}, are isolated poles and on or to the left of $x = x_o$. If $x_1 > x_o$ is a constant then all $\left\{s_k\right\}$ must be to the left of

$$\gamma_n: \left\{ s \middle| s = x_1 + iy,\ -A_n \leqslant y \leqslant A_n \right\}.$$

The indices may be assigned to the poles such that $d\left[s_k, \gamma_n\right] \leqslant d\left[s_{k+1}, \gamma_n\right]$ and $\left|\mathcal{G}\left[s_k\right]\right| \leqslant \left|\mathcal{G}\left[s_{k+1}\right]\right|$. Now define an infinite sequence of contours $\left\{c_n\right\}$ n = 1,2,3,... such that

$$c_n: \left\{ s \middle| s - x_1 + i0 = R_n(\theta)\, e^{i\theta},\ \frac{\pi}{2} \leqslant \theta \leqslant \frac{3\pi}{2};\ R_n(\phi_n) > \left|s_n - x_1\right|, \right.$$

$$\left. \phi_n = \arg(s_n - x_1) \text{ and } R(\tfrac{\pi}{2}) = R(\tfrac{3\pi}{2}) = A_n;\ d\left[s_k, c_n\right] > \epsilon > 0, \right\}.$$

Thus C_n and γ_n form a closed contour with singular points $\{s_k\}$ $k = 1,2,3,\ldots, n$ in its interior. It is also to be required that

$$\lim_{n\to\infty} \int_{C_n} F(s)\, e^{st}\, ds = 0.$$

Next, consider the residue theorem.

Theorem 2.4.1. Let C be a simple, closed, rectifiable, oriented curve and let f(s) be a function that is holomorphic at all points on this contour and its interior except for a finite number of isolated singular points $\{s_k\}$ $k = 1,2,3,\ldots N$. Then,

$$\frac{1}{2\pi i} \oint_C f(s)\, ds = \sum_{k=1}^{N} \text{Res}\,[f,\ s_k]$$

where Res $[f,\ s_k]$ is the residue of f at s_k. For a proof see [7, pp. 241-42].

If this theorem is applied to a function $F(s)\, e^{st}$ which satisfies the preceding requirements then,

$$\frac{1}{2\pi i} \oint_C F(s)\, e^{st}\, ds = \frac{1}{2\pi i}\left(\int_{C_n} F(s)\, e^{st}\, ds + \int_{\gamma_n} F(s)\, e^{st}\, ds \right)$$

$$= \sum_{k=1}^{N} \text{Res}\,[F(s)\, e^{st},\ s_k].$$

If one lets n increase without bound then,

$$\frac{1}{2\pi i} \int_{\gamma_n} F(s)\, e^{st}\, ds = \frac{1}{2\pi i} \lim_{A_n\to\infty} \int_{x_1 - iA_n}^{x_1 + iA_n} F(s)\, e^{st}\, ds$$

$$= \sum_{k=1}^{N} \text{Res} \left[F(s)\, e^{st},\ s_k \right].$$

If the number of singular points of F(s) is finite then

$$f(t) = \sum_{k=1}^{N} \text{Res} \left[F(s)\, e^{st},\ s_k \right].$$

If there is an infinite number of singular points, each contour of the infinite sequence of contours will contain at most a finite number of poles not contained in those of lower index. This leads to an infinite series of residues. If this series converges then f(t) exists and is equal to the sum of the series.

The technique, that was just outlined, will be demonstrated for the class of rational functions for which the degree of the numerator is less than that of the denominator. In this way, a form of the Heaviside expansion theorem will be developed.

Let C_n of the previous discussion be composed of the four contours

$$C_{n_1}: \left\{ s \,\middle|\, s = x + iR_n,\ x_1 \geqslant x \geqslant 0 \right\}$$

$$C_{n_2}: \left\{ s \,\middle|\, s = R_n e^{i\theta},\ \frac{\pi}{2} \leqslant \theta \leqslant \pi \right\}$$

$$C_{n_3}: \left\{ s \,\middle|\, s = R_n e^{i\theta}, \pi \leqslant \theta \leqslant \frac{3\pi}{2} \right\}$$

$$C_{n_4}: \left\{ s \,\middle|\, s = x + iR_n,\ 0 \leqslant x \leqslant x_1 \right\}.$$

Let the contour just given be the C_n in the following lemma.

Lemma 3.4.1. If F(s) is $0\left[\frac{1}{s^k}\right]$ as s becomes infinite, then

$$\lim_{R_n\to\infty}\int_{C_n} F(s)\, e^{st}\, ds = 0 \text{ for } k>0 \text{ and } t>0.$$

Proof. From the proof of theorem 2.3.7 it follows that

$$\lim_{R_n\to\infty}\int_{C_{n_1}} F(s)\, e^{st}\, ds = \lim_{R\to\infty}\int_{C_{n_4}} F(s)\, e^{st}\, ds = 0.$$

By hypothesis $|F(s)| < \frac{M}{|s|^k}$ where M is a constant.

Hence

$$\left|\int_{C_{n_2}} F(s)\, e^{st}\, ds\right| \leqslant \int_{C_{n_2}} |F(s)|\, e^{xt}\, |ds| \leqslant M\, R_n^{1-k}\int_{\pi/2}^{\pi} e^{(R_n\cos\theta)t}\, d\theta$$

since $|ds| = R_n d\theta$ and $x = R_n \cos\theta$ on the contour C_{n_2}.

But cos $= -\sin(\theta - \frac{\pi}{2})$; therefore,

$$\left|\int_{C_{n_2}} F(s)\, e^{st}\, ds\right| \leqslant M\, R_n^{1-k}\int_{0}^{\pi/2} e^{-R_n t\sin\theta}\, d\theta .$$

Also,

$$\left|\int_{C_{n_3}} F(s)\, e^{st}\, ds\right| \leqslant M\, R_n^{1-k}\int_{\pi}^{3\pi/2} e^{R_n t\cos\theta}\, d\theta$$

$$= M\, R_n^{1-k}\int_{-\pi/2}^{0} e^{R_n t\sin\theta}\, d\theta = M\, R_n^{1-k}\int_{0}^{\pi/2} e^{-R_n t\sin\theta}\, d\theta .$$

This is true since,

$$\text{Cos}\,\theta = \sin(\theta - \frac{3\pi}{2}) \text{ and } \sin(-\theta) = -\sin\theta .$$

But since $\sin\theta \geqslant \frac{2\theta}{\pi}$ for $0 \leqslant \theta \leqslant \frac{\pi}{2}$ then,

$$M\,R_n^{1-k}\int_0^{\pi/2} e^{-R_n t \sin\theta}\,d\theta \leqslant M\,R_n^{1-k}\int_0^{\pi/2} e^{\frac{-2R_n\theta t}{\pi}}\,d\theta$$

$$= -\frac{M\pi}{2tR_n^k}\left(e^{-R_n t} - 1\right) < \frac{M\pi}{2tR_n^k}.$$

Therefore,

$$\lim_{R_n\to\infty}\left|\int_{C_{n_2}} F(s)\,e^{st}\,ds\right| = \lim_{R_n\to\infty}\left|\int_{C_{n_3}} F(s)\,e^{st}\,ds\right| = \lim_{R_n\to\infty}\frac{M\pi}{2tR_n^k} = 0$$

for $k > 0$ and $t > 0$. Hence,

$$\lim_{R_n\to\infty}\left|\int_{C_n} F(s)\,e^{st}\,ds\right| = 0.$$

This proves the lemma.

Recall that any rational function may be expressed as the quotient of two polynomials. By the fundamental theorem of algebra

$$F(s) = \frac{P(s)}{Q(s)} = \frac{P(s)}{\prod_{k=1}^{N}(s - s_k)^{m_k}}.$$

The singular points of $F(s)\,e^{st}$ will be the points $\{s_k\}$. Let s_N be the singular point of greatest absolute value, hence all $\{s_k\}$ will be enclosed in the circle $s = R\,e^{i\theta}$ where $R > r_0 = |s_N|$. Outside of the region, $F(s)$ will be bounded since the degree of $Q(s)$ is greater than the degree of $P(s)$;

in fact, there is an M such that $\left|F(s)\right| \leqslant \frac{M}{|s|}$. Thus lemma 3.4.1 applies. Applying the residue theorem it follows that

$$f(t) = \frac{1}{2\pi i} \lim_{R\to\infty} \int_{x_1 - iR}^{x_1 + iR} F(s)\, e^{st}\, ds = \sum_{k=1}^{N} \operatorname{Res} \left[F(s)\, e^{st},\ s_k\right]$$

where

$$\operatorname{Res} \left[F(s)\, e^{st},\ s_k\right]$$

$$= \frac{1}{(m_k - 1)!} \frac{\partial^{m_k - 1}}{\partial s^{m_k - 1}} \left[(s - s_k)^{m_k} \frac{e^{s_k t}\, P(s_k)}{\prod_{k=1}^{N} (s - s_k)^{m_k}} \right]$$

for $t > 0$ and $x_o > \mathcal{R}\left[s_1\right]$ where s_1 is the pole with maximum real component. If $m = 1$, then

$$\operatorname{Res} \left[F(s)\, e^{st},\ s_k\right] = e^{s_k t} \lim_{s \to s_k} \frac{(s - s_k)\, P(s)}{Q(s)}$$

$$= e^{s_k t} \frac{P(s_k)}{\prod_{j \neq k} (s - s_j)} = \frac{e^{s_k t}\, P(s_k)}{Q'\,(s_k)}, \quad Q' = \frac{dQ}{ds}$$

thus if $M = 1$ for all k, then

$$f(t) = \sum_{k=1}^{N} \frac{e^{s_k t}\, P(s_k)}{Q'\,(s_k)}, \text{ for } t > 0.$$

The following form of the Heaviside expansion theorem has been proven.

<u>Theorem 3.4.2</u>. Let $F(s) = \frac{P(s)}{Q(s)}$ be a rational function

where the degree of P(s) is less than the degree of Q(s) and if Q(s) has no repeated factors then

$$\mathcal{L}^{-1}\left\{F(s)\right\} = \sum_{k=1}^{N} e^{s_k t} \frac{P(s_k)}{Q'(s_k)} = f(t), \ t > 0$$

where the $\left\{s_k\right\}$ are the coordinates of the singular points of F(s).

CONCLUDING REMARKS

The Laplace transform has been treated as a special case of the exponential form of the Fourier integral. From this point of view the properties of the Laplace transform may be derived from those of the Fourier transform. The Fourier integral theorem expresses a dual relationship between a class of object functions and their transform images. It was from the Fourier inversion formula that the complex form of the inversion integral of the Laplace transform was then derived. The Laplace transform was constructed by replacing the object function of the Fourier transform by a new object function multiplied by an exponential convergence factor e^{-xt} where x is a real variable. This choice not only broadened the scope of the transform process to include all real APC functions of exponential order but lead naturally into the introduction of the theory of functions of a complex variable. The utilization of this latter theory not only made it possible to establish conditions of uniqueness for the inversion integral but also provided a practical approach to its evaluation in terms of the calculus of residues. Finally, the Heaviside expansion theorem was derived as an example. Aside from its utility in demonstrating the theory, this example was chosen because of its historical value. A form of this theorem was the basis of the original operational calculus and Heaviside's failure to place it on a rigorous base was the source of the controversy noted earlier.

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THE INVERSION INTEGRAL OF THE LAPLACE TRANSFORM

by

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The central purpose of the paper is to outline the mathematical development of the inversion integral of the Laplace transform. An attempt is made in the introduction to place the subject matter in context by defining the Laplace transform and by giving a brief historical development of the operational calculus. Also, a simple example is given to relate the modern operational approach to the solution of differential equations by classical methods.

The theory of the Laplace transform and its inversion integral is approached via the exponential form of the Fourier integral theorem. It is proven for absolutely integrable almost piecewise continuous functions. The Laplace transform was constructed by replacing the object function in the Fourier transform with $f(t)\ e^{-xt}$ where $f(t) = 0,\ t < 0$. Thus, the Laplace transform is given as

$$\mathcal{L}\left\{f(t)\right\} = \int_{-\infty}^{\infty} f(t)\ e^{-xt}\ e^{-iyt}\ dt = \int_{0}^{\infty} f(t)\ e^{-st}\ dt = F(s)$$

where $s = x + iy$ is a complex variable. The inversion integral is formally derived by substituting into the inversion formula of the Fourier integral as follows:

$$f(t)\ e^{-xt} = \frac{1}{2\pi}\, PV \int_{-\infty}^{\infty} F(x + iy)\ e^{iyt}\ dy$$

where $F(s)$ is the function to be inverted. By considering x as a constant and choosing as a contour the vertical line $s = x$, one obtains

$$f(t) = \frac{1}{2\pi i} \lim_{A\to\infty} \int_{x-iA}^{x+iA} F(s)\ e^{st}\ dt$$

as the inversion integral. This formal procedure introduces ambiguity in terms of what values of x should be used in evaluating the contour integral.

To remove this ambiguity the convergence of the Laplace integral is investigated. It is found that if f(t) is of exponential order x_0 that the Laplace integral will converge for $x > x_0$. It is also found that the Laplace transform F(s) is a holomorphic function in the region of convergence of the integral. It is then shown that if F(s) is of order s^{-k}, $k > 0$ or if F(s) is the Laplace transform of an almost piecewise continuous function of exponential order x_0, the value of the inversion integral is independent of x for $x > x_0$. Thus the ambiguity is removed for this class of functions. A theorem is developed for the evaluation of the inversion integral by the calculus of residues. The Heaviside expansion theorem is developed to demonstrate the use of these techniques.